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Conformal coupling of the scalar field with gravity in higher dimensions and invariant powers of the Laplacian

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Abstract

The hierarchy of conformally coupled scalars with the increasing scaling dimensions $\Delta_k = k - d/2$, $k = 1, 2, 3, \ldots$ connected with the k-th Euler density in the corresponding space-time dimensions $d \geq 2k$ is proposed. The corresponding conformal invariant Lagrangian with the k-th power of Laplacian for the already known cases $k = 1, 2$ is reviewed, and the subsequent case of $k = 3$ is completely constructed and analyzed.

1 Introduction

Our aim in this work is the construction of a hierarchy of conformally invariant Lagrangians in spacetime dimensions $d \geq 2k$, describing the nonminimal coupling of gravity with a scalar field whose conformal dimension is $\Delta_{(k)} = k - d/2$. A remarkable feature of these systems is the appearance of the k-th Euler density $E_{(k)}$ in the k-th member of this hierarchy. The $k = 1, 2$ cases are known, and here we supply the $k = 3$ case concretely, suggesting the arbitrary k case.

The conformal coupling of a scalar field with gravity in different dimensions has been a subject of interest in quantum field theory in curved spacetimes [1]. In recent years it has attracted special attention in the context of new developments in the area of AdS/CFT [2] correspondence, and in investigations of higher order and higher spin gravitating systems in general [3]. Conformally invariant field theories in higher dimensions are particularly interesting because they present a universal tool for investigations of their quantum properties, such as conformal or trace anomalies [4]. Another important properety of conformally invariant theories in arbitrary dimensions is, that the method of dimensional regularisation can be employed as a conformally invariant regularisation in higher dimensions for the construction of anomalous effective actions [5]. Note also that in connection with higher spin gauge field interactions with a scalar field, this coupling and Weyl invariance itself, can be generalized [6].

In this article we propose a hierarchy of such couplings of gravity to scalar fields with increasing scaling dimensions parameterized by a natural number k, and living in all space-time dimensions $d \geq 2k$. Actually this hierarchy corresponds to the conformally invariant k-th power of the Laplacian acting on a scalar field with conformal dimension $\Delta_{(k)} = k - d/2$, in spacetime dimensions $d \geq 2k$. From the other hand we propose the connection between this hierarchy and the k-th Euler density $E_{(k)}$ lifted to spacetime dimensions greater than $2k$. For completeness, we verify this proposal in the well known text book case of $k = 1$ [1]. We then turn to the known case in $d = 4$ [7, 8], and the fourth order conformally covariant operator in dimension $d \geq 4$ obtained in [9, 10] long ago, which provides us with a further check of our proposal, now involving the second Euler density $E_{(2)}$. Finally in the last section we perform the new calculation of the locally Weyl invariant third power of the Laplacian in spacetime dimensions $d \geq 6$, or in another words we construct a conformally invariant action for the scalar with conformal dimension $3 - d/2$ coupled with gravity. In all three cases we have found the corresponding Euler density $E_{(k)}$ as part of the invariant action, proportional to the first order of $\Delta_{(k)}$, and without derivatives. Taking into account the rather technical character of this article we devote a substantial section, Section **2**, with a more or less complete technical setup and all the formulas which we have used in our calculations.

2 Notations and Conventions

We work in a d dimensional curved space and use the following conventions for covariant derivatives and curvatures:

$$
\begin{aligned}
\nabla_\mu V_\lambda^\rho &= \partial_\mu V_\lambda^\rho + \Gamma_{\mu\sigma}^\rho V_\lambda^\sigma - \Gamma_{\mu\lambda}^\sigma V_\sigma^\rho, && (1)\\
\Gamma_{\mu\nu}^\rho &= \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right), && (2)\\
\left[\nabla_\mu, \nabla_\nu\right] V_\lambda^\rho &= R_{\mu\nu\sigma}{}^\rho V_\lambda^\sigma - R_{\mu\nu\lambda}{}^\sigma V_\sigma^\rho, && (3)\\
R_{\mu\nu\lambda}{}^\rho &= \partial_\mu \Gamma_{\nu\lambda}^\rho - \partial_\nu \Gamma_{\mu\lambda}^\rho + \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\lambda}^\sigma - \Gamma_{\nu\sigma}^\rho \Gamma_{\mu\lambda}^\sigma, && (4)\\
R_{\mu\lambda} &= R_{\mu\rho\lambda}{}^\rho \quad , \quad R = R_\mu^\mu. && (5)
\end{aligned}
$$

The corresponding local conformal transformations (Weyl rescalings)

$$
\begin{aligned}
\delta g_{\mu\nu} &= 2\sigma(x) g_{\mu\nu}, \qquad \delta g^{\mu\nu} = -2\sigma(x) g^{\mu\nu}, && (6)\\
\delta \Gamma_{\mu\nu}^\lambda &= \partial_\mu \sigma \delta_\nu^\lambda + \partial_\nu \sigma \delta_\mu^\lambda - g_{\mu\nu} \partial^\lambda \sigma, && (7)\\
\delta R_{\mu\nu\lambda}{}^\rho &= \nabla_\mu \partial_\lambda \sigma \delta_\nu^\rho - \nabla_\nu \partial_\lambda \sigma \delta_\mu^\rho + g_{\mu\lambda} \nabla_\nu \partial^\rho \sigma - g_{\nu\lambda} \nabla_\mu \partial^\rho \sigma, && (8)\\
\delta R_{\mu\lambda} &= (d-2) \nabla_\mu \partial_\lambda \sigma + g_{\mu\lambda} \Box \sigma, && (9)\\
\delta R &= -2\sigma R + 2(d-1) \Box \sigma\,, && (10)
\end{aligned}
$$

are first oder in the infinitesimal local scaling parameter σ.

We then introduce the Weyl (W) and Schouten (K) tensors, as well as the scalar J

$$
\begin{aligned}
R_{\mu\nu} &= (d-2) K_{\mu\nu} + g_{\mu\nu} J, \quad J = \frac{1}{2(d-1)} R\ , && (11)\\
W_{\mu\nu\lambda}{}^\rho &= R_{\mu\nu\lambda}^\rho - K_{\mu\lambda} \delta_\nu^\rho + K_{\nu\lambda} \delta_\mu^\rho - K_\nu^\rho g_{\mu\lambda} + K_\mu^\rho g_{\nu\lambda}\ , && (12)\\
\delta K_{\mu\nu} &= \nabla_\mu \partial_\nu \sigma\,, && (13)\\
\delta J &= -2\sigma J + \Box \sigma\,, && (14)\\
\delta W_{\mu\nu\lambda}{}^\rho &= 0\,, && (15)
\end{aligned}
$$

which are more convenient because their conformal transformations are "diagonal".

To describe the Bianchi identity with these tensors, we have to introduce the so called Cotton tensor

$$
\begin{aligned}
C_{\mu\nu\lambda} &= \nabla_\mu K_{\nu\lambda} - \nabla_\nu K_{\mu\lambda}\ , && (16)\\
\delta C_{\mu\nu\lambda} &= -\partial_\alpha \sigma W_{\mu\nu\lambda}{}^\alpha \quad , \quad C_{[\mu\nu\lambda]=0}\,. && (17)
\end{aligned}
$$

All important properties of these tensors following from the Bianchi identity can then be listed as

$$
\begin{aligned}
\nabla_{[\alpha} W_{\mu\nu]\lambda}{}^\rho &= g_{\lambda[\alpha} C_{\mu\nu]}{}^\rho - \delta_{[\alpha}^\rho C_{\mu\nu]\lambda}\ , && (18)\\
\nabla_\alpha W_{\mu\nu\lambda}{}^\alpha &= (3-d)\, C_{\mu\nu\lambda}\ , && (19)\\
\nabla^\mu K_{\mu\nu} &= \partial_\nu J\,, && (20)\\
C_{\mu\nu}{}^\nu &= 0\ , \qquad \nabla^\lambda C_{\mu\nu\lambda} = 0\,. && (21)
\end{aligned}
$$

Finally we introduce the last important conformal object in the above listed hierarchy, namely the symmetric and traceless Bach tensor

$$B_{\mu\nu} \;=\; \nabla^{\lambda} C_{\lambda\mu\nu} + K^{\lambda}_{\alpha} W_{\lambda\mu\nu}{}^{\alpha}, \tag{22}$$

whose conformal transformation and divergence are expressed in terms of the Cotton and the Schouten tensors as follows

$$\begin{aligned} \delta B_{\mu\nu} &= (d-4)\nabla^{\lambda}\sigma\left(C_{\lambda\mu\nu} + C_{\lambda\nu\mu}\right), && (23)\\ \nabla^{\mu} B_{\mu\nu} &= (d-4) C_{\alpha\nu\beta} K^{\alpha\beta}. && (24) \end{aligned}$$

Note that only in four dimensions is the Bach tensor conformally invariant and divergenceless.

This basis of B, C, K, J, W tensors is all we need to construct any conformally invariant object in arbitrary dimensions. Finally for any scalar $f^{\Delta}(x)$ with arbitrary scaling dimension Δ we can easily derive the following important relations

$$\begin{aligned} \delta\left(\nabla_{\mu}\partial_{\nu} f^{\Delta}\right) &= \Delta\sigma\nabla_{\mu}\partial_{\nu} f^{\Delta} + \Delta f^{\Delta}\nabla_{\mu}\partial_{\nu}\sigma + (\Delta-1)\partial_{(\mu}\sigma\partial_{\nu)} f^{\Delta} + g_{\mu\nu}\partial^{\lambda}\sigma\partial_{\lambda} f^{\Delta}, && (25)\\ \delta\left(\Box f^{\Delta}\right) &= (\Delta-2)\sigma\Box f^{\Delta} + \Delta f^{\Delta}\Box\sigma + (d+2\Delta-2)\partial^{\lambda}\sigma\partial_{\lambda} f^{\Delta}, && (26) \end{aligned}$$

by using the transformation (7) for Christoffel symbols.

3 Hierarchies of conformal scalars and Euler densities

In this section we introduce the hierarchy of scalar fields $\varphi_{(k)}$, where $k = 1, 2, 3, \ldots$ with the corresponding scaling dimensions and infinitesimal conformal transformations

$$\begin{aligned} \Delta_{(k)} &= k - d/2\,, && (27)\\ \delta\varphi_{(k)} : &= \Delta_{(k)}\sigma\varphi_{(k)}. && (28) \end{aligned}$$

Each of these exist in the spacetime dimensions $d \geq 2k$, and with the minimal dimension vanishing, $\Delta_{(k)} = 0$ when $d = 2k$.

Let us now introduce the hierarchy of the Euler densities *

$$E_{(k)} := \frac{1}{2k(d-2k)!}\delta^{\alpha_1\ldots\alpha_{d-2k}\mu_1\mu_2\ldots\mu_{2k-1}\mu_{2k}}_{\alpha_1\ldots\alpha_{d-2k}\nu_1\nu_2\ldots\nu_{2k-1}\nu_{2k}} R^{\nu_1\nu_2}_{\mu_1\mu_2}\ldots R^{\nu_{2k-1}\nu_{2k}}_{\mu_{2k-1}\mu_{2k}}. \tag{29}$$

This set of objects exist as Lagrangians in space time dimensions $d \geq 2k$, but for the minimal case $d = 2k$, E_k is a total divergenece such that its integral is a

*Note that the usual Einstein–Hilbert Lagrangian in d dimensions is the $k = 1$ member of this hierarchy of gravitational Lagrangians.

topological invariant, the Euler characteristic. In these dimensions E_k trivialize as Lagrangians but describe the topological part of the trace anomaly in the corresponding even space-time dimension $2k$.

The idea of this article is the following observation: *There should be a one to one correspondence between the conformally coupled scalars $\varphi_{(k)}$ and the Euler densities $E_{(k)}$.*

Our first step in proving this is to start from the action of the well known non minimal conformally coupled scalar in the space-time dimension d and with conformal dimension $\Delta_1 = 1 - d/2$

$$S_{(1)} = \frac{1}{2}\int d^dx\sqrt{g}\left\{g^{\mu\nu}\partial_\mu\varphi_{(1)}\partial_\nu\varphi_{(1)} - \frac{d-2}{4(d-1)}R\varphi_{(1)}^2\right\}. \qquad (30)$$

We first see that the second term without derivatives and proportional to the scaling dimension can be written in the form $-\frac{d-2}{4(d-1)}R = \Delta_{(1)}J$. After that the proof of the conformal invariance of the action (30) becomes trivial: We write (26) for $\Delta = \Delta_{(1)}$ and use (14), from which it follows that $\delta\left[\sqrt{g}\varphi_{(1)}\left(\Box - \Delta_{(1)}J\right)\varphi_{(1)}\right] = 0$. We next evaluate (29) for $k = 1$

$$E_{(1)} = 2(d-1)J\,. \qquad (31)$$

Finally we see that (30) can be rewritten in the form

$$S_{(1)} = \frac{1}{2}\int d^dx\sqrt{g}\left\{-\varphi_{(1)}\Box\varphi_{(1)} + \frac{\Delta_{(1)}}{2(d-1)}E_{(1)}\varphi_{(1)}^2\right\}\,. \qquad (32)$$

We now see that *derivative independent part of the conformally invariant action is proportional to the scaling dimension times the first Euler density*. Note again that both objects degenerate in minimal dimension $d = 2$ where the Laplacian itself is conformally invariant and the Euler density describes the topological invariant, which is the two dimensional trace anomaly.

The next step in our considerations is the $k = 2$ case. Again this higher derivative action (or 4-th order conformal invariant operator) is known since many years [7, 8] for dimension 4 as well as for general d [9, 10]. All this is presented in [11] where many of the invariant objects are considered. In our work, we rederived this Lagrangian just applying the Noether procedure to the local conformal variation of the following suitable object

$$S^0_{(2)} = \frac{1}{2}\int d^dx\sqrt{g}\left(\widehat{D}_{(2)}\varphi_{(2)}\right)^2, \qquad (33)$$

whose Weyl transformation includes only the first derivatives of the parameter.

In (33) and thereafter, we use the notation

$$\begin{aligned}
\widehat{D}_{(k)} &:= \Box - \Delta_{(k)} J, \quad k = 1,2,3,\dots\ , && (34)\\
\delta\left(\widehat{D}_{(k)}\varphi_{(k)}\right) &= (\Delta_{(k)}-2)\widehat{D}_{(k)}\varphi_{(k)} + 2(k-1)\partial^\mu\sigma\partial_\mu\varphi_{(k)}, && (35)\\
\widehat{D}^{(k)}_{\mu\nu} &:= \nabla_\mu\partial_\nu - \Delta_{(k)}K_{\mu\nu}\,, \quad g^{\mu\nu}\widehat{D}^{(k)}_{\mu\nu} = \widehat{D}_{(k)}\,, && (36)\\
\delta\left(\widehat{D}^{(k)}_{\mu\nu}\varphi_{(k)}\right) &= \Delta_{(k)}\sigma\widehat{D}^{(k)}_{\mu\nu}\varphi_{(k)} + (\Delta_{(k)}-1)\partial_{(\mu}\sigma\partial_{\nu)}\varphi_{(k)} + g_{\mu\nu}\partial^\lambda\sigma\partial_\lambda\varphi_{(k)}. && (37)
\end{aligned}$$

Performing the functional integration of the Weyl variation of the (33) is now just a matter of some partial integration, elimination of the second derivatives of σ using (13),(14) and cancelation of terms linear in $\partial\sigma$ using the Bianchi identity (20). It should be noted here that all these types of calculations could instead be performed using the powerful method proposed in [12]. Here we presented only the direct Noether procedure because that will be more suitable for us in the next section. After all these manipulations we arrive at the following action

$$\begin{aligned}
S^1_{(2)} &= \frac{1}{2}\int d^dx\sqrt{g}\Big\{\left(\widehat{D}_{(2)}\varphi_{(2)}\right)^2 + 4K^{\mu\nu}\partial_\mu\varphi_{(2)}\partial_\nu\varphi_{(2)} - 2J\partial^\mu\varphi_{(2)}\partial_\mu\varphi_{(2)}\\
&\quad +2\Delta_{(2)}\left(K^2-J^2\right)\varphi^2_{(2)}\Big\} && (38)
\end{aligned}$$

Then after some work we can evaluate $E_{(2)}$ using (29) and (12) to be

$$E_{(2)} = W^2 - 4(d-3)(d-2)\left(K^2-J^2\right)\,. \qquad (39)$$

W see that the $\varphi^2_{(2)}$ term in (38) which is linear in $\Delta_{(2)}$, is proportional to the Weyl tensor independent part of the Euler density. The other term without derivatives is proportional to $\Delta^2_{(2)}$. This noninvariant part of the four dimensional trace anomaly arises in AdS/CFT [13] and carries the name "holographic", and corresponds to the maximally supersymmetric gauge theory on the boundary of AdS_4.

The combination

$$-\frac{1}{2}\int d^dx\sqrt{g}\left\{\frac{\Delta_2}{2(d-3)(d-2)}W^2\varphi^2_{(2)}\right\}, \qquad (40)$$

on the other hand is also conformally invariant and can be added to (38) at no cost. This leads us to our final result

$$\begin{aligned}
S^E_{(2)} &= \frac{1}{2}\int d^dx\sqrt{g}\,\{\varphi_{(2)}\Box^2\varphi_{(2)} - 2\Delta_{(2)}J\varphi_{(2)}\Box\varphi_{(2)} + \Delta^2_{(2)}J^2\varphi^2_{(2)}\\
&- \quad 2J\partial^\mu\varphi_{(2)}\partial_\mu\varphi_{(2)} + 4K^{\mu\nu}\partial_\mu\varphi_{(2)}\partial_\nu\varphi_{(2)} - \frac{\Delta_{(2)}}{2(d-3)(d-2)}E_{(2)}\varphi^2_{(2)}\Big\}, && (41)
\end{aligned}$$

confirming our main observation in the $k=2$ case.

4 The $\Delta_3 = 3 - d/2$ case

To confirm our main observation, verified for $k = 1, 2$ above, and present it as an assertion for general k, we need to carry out this verification in the next nontrivial case of $k = 3$. This is the content of the present Section, which consists of the explicit calculation of the conformally invariant action analogous to (32) and (41) for $k = 1, 2$. In this case we will follow again the same strategy.

Taking into account that $\widehat{D}_{(3)}\varphi_{(3)}$ scales as an object with the dimension $\Delta_{(1)} = \Delta_{(3)} - 2$ we start from the following initial Lagrangian

$$S^0_{(3)} = -\frac{1}{2}\int d^dx\sqrt{g}\left\{\widehat{D}_{(3)}\varphi_{(3)}\left(\widehat{D}_{(3)} + 2J\right)\widehat{D}_{(3)}\varphi_{(3)}\right\}, \tag{42}$$

with the more or less simple Weyl variation

$$\begin{aligned}\delta S^0_{(3)} = -\int d^dx\sqrt{g}\Big\{&4\widehat{D}_{(3)}\varphi_{(3)}\left(\Delta_{(3)}\varphi_{(3)}\partial^\lambda\sigma\partial_\lambda J + 4(\Delta_{(3)} - 2)K^{\mu\nu}\partial_\mu\sigma\partial_\nu\varphi_{(3)}\right)\\ -2\widehat{D}_{(3)}\varphi_{(3)}&\left(\widehat{D}_{(3)}\varphi_{(3)}\delta J - 4\widehat{D}^{(3)}_{\mu\nu}\varphi_{(3)}\delta K^{\mu\nu} - 2\partial_\lambda\varphi_{(3)}\partial^\lambda\delta J - 2\Delta_{(3)}\delta(K^2)\varphi_{(3)}\right)\Big\}. \end{aligned}\tag{43}$$

The second line in (43) can be integrated adding to the $S^0_{(3)}$ the following terms

$$\begin{aligned}S^1_{(3)} \;=\; -\int d^dx\sqrt{g}\Big\{&2(\widehat{D}_{(3)}\varphi_{(3)})^2 J - 8\widehat{D}_{(3)}\varphi_{(3)}\widehat{D}^{(3)}_{\mu\nu}\varphi_{(3)}K^{\mu\nu}\\ &-4\widehat{D}_{(3)}\varphi_{(3)}\partial_\lambda\varphi_{(3)}\partial^\lambda J - 4\Delta_{(3)}\widehat{D}_{(3)}\varphi_{(3)}K^2\varphi_{(3)}\Big\}.\end{aligned}\tag{44}$$

Writing the variation of the $S^{0+1}_{(3)}$ is rather more complicated. First we should separate the Laplacians from $\Delta_{(3)}J$ in the terms with $\widehat{D}_{(3)}\varphi_{(3)}$, then, performing some partial integrations we redistribute derivatives and separate the terms $\partial_\mu\varphi_{(3)}\partial_\nu\varphi_{(3)}$, $\partial_\mu\varphi_{(3)}\partial^\mu\varphi_{(3)}$ and $\varphi^2_{(3)}$, that are irreducible under partial integration . After some manipulations, using (16) and Bianchi identities, we obtain

$$\begin{aligned}&\delta S^0_{(3)} + \delta S^1_{(3)} = -\delta S^2_{(3)} - \delta S^{\Delta_{(3)}}_{(3)}\\ &+\int d^dx\sqrt{g}\left\{16C^{\lambda\mu\nu}\partial_\lambda\sigma\partial_\mu\varphi_{(3)}\partial_\nu\varphi_{(3)} + 24\Delta_{(3)}C^{\lambda\mu\nu}\partial_\lambda\sigma K_{\mu\nu}\varphi^2_{(3)}\right\},\end{aligned}\tag{45}$$

where

$$S^2_{(3)} = \int d^dx\sqrt{g}\left\{24K^{2\mu\nu} - 16JK^{\mu\nu} - 4K^2g^{\mu\nu}\right\}\partial_\mu\varphi_{(3)}\partial_\nu\varphi_{(3)}, \tag{46}$$

$$S^{\Delta_{(3)}}_{(3)} = 4\Delta_{(3)}\int d^dx\sqrt{g}\left\{J^3 - 3K^2J + 2K^3\right\}\varphi^2_{(3)}. \tag{47}$$

Now to cancel the second line in (45) with the Cotton tensor we have to turn to the Bach tensor transformation (23). It is easy to see that the following combination

$$S^B_{(3)} = -\frac{8}{d-4}\int d^dx\sqrt{g}\left\{B^{\mu\nu}\partial_\mu\varphi_{(3)}\partial_\nu\varphi_{(3)} + \Delta_{(3)}B^{\mu\nu}K_{\mu\nu}\varphi^2_{(3)}\right\}, \tag{48}$$

make our action completely locally conformal invariant. It follows that that the required locally Weyl invariant action for the $k = 3$ case is

$$S_{(3)} = \sum_{i=0}^{2} S^i_{(3)} + S^{\Delta_{(3)}}_{(3)} + S^B_{(3)}. \tag{49}$$

Now we analyze the linear on $\Delta_{(3)}\varphi^2_{(3)}$ part of (49):

$$4\Delta_{(3)} \int d^dx\sqrt{g}\left\{J^3 - 3K^2J + 2K^3 - \frac{2}{d-2}B^{\mu\nu}K_{\mu\nu}\right\}\varphi^2_{(3)}. \tag{50}$$

We see again that this part coincides with the so-called "holographic" anomaly [13] in 6 dimensions written in general spacetime dimension d (see also [16] for the role of the Bach tensor in holography). The main property of the holographic anomaly is that it is a special combination of the Euler density with the other three Weyl invariants [14] which reduce the topological part of the anomaly to the expression (50) (see [15] for the correct separation), which is zero for the Ricci flat metric.

But this is for the anomaly itself in $d = 6$. Here we are concerned with the invariant Lagrangian and presence of the scalar field and the integral make our considerations easier. To get the invariant action with the whole third Euler density, we have to perform some more work, and find that there is another invariant action with the maximum of four derivatives. This action can be obtained, using the same Noether procedure, to render the following initial term

$$S^0_W = \frac{8}{(d-3)(d-4)}\int d^dx\sqrt{g}W^{\mu\alpha\nu\beta}\widehat{D}^{(3)}_{\mu\nu}\varphi_{(3)}\widehat{D}^{(3)}_{\alpha\beta}\varphi_{(3)} \tag{51}$$

invariant. After some lengthy but straightforward calculation we arrive at the following locally conformal invariant action.

$$S_W = S^B_{(3)} - S^0_W - S^1_W - S^{\Delta_{(3)}}_W, \quad \delta S_W = 0, \tag{52}$$

where

$$S^1_W = \int d^dx\sqrt{g}\left\{\frac{16W^{\mu\alpha\nu\beta}K_{\alpha\beta}}{(d-3)} + \frac{3W^2g^{\mu\nu} - 12W^{2\mu\nu}}{(d-3)(d-4)}\right\}\partial_\mu\varphi_{(3)}\partial_\nu\varphi_{(3)}, \tag{53}$$

$$S^{\Delta_{(3)}}_W = \Delta_{(3)}\int d^dx\sqrt{g}\left\{\frac{12W^{\mu\alpha\nu\beta}K_{\mu\nu}K_{\alpha\beta}}{(d-3)} + \frac{3W^2J - 12W^{2\mu\nu}K_{\mu\nu}}{(d-3)(d-4)}\right\}\varphi^2_{(3)}. \tag{54}$$

To derive this we used the Bianchi identity (18) contracted with the Weyl tensor. This leads to the following relation

$$\frac{1}{2}\partial_\alpha W^2 - 2\nabla_\mu W^{2\mu}_\alpha = 2(d-4)C_{\lambda\rho}{}^{\nu}W_{\nu\alpha}{}^{\lambda\rho}, \tag{55}$$

which generates the terms quadratic in the Weyl tensor in (52)-(54). Therefore the existence of the invariant (52) allows us to replace the Bach tensor dependent term $S^{B}_{(3)}$ in (49) with W dependent terms and obtain

$$S^{\mathcal{A}}_{(3)} = \sum_{i=0}^{2} S^{i}_{(3)} + S^{0}_{W} + S^{1}_{W} + S^{\Delta_{(3)}}_{(3)} + S^{\Delta_{(3)}}_{W}. \tag{56}$$

Then we see that all terms proportional to $\Delta_3\varphi^2_{(3)}$ are accumulated in the last two terms of (56)

$$S^{\Delta_{(3)}}_{(3)} + S^{\Delta_{(3)}}_{W} = \frac{3\Delta_3}{(d-5)(d-4)(d-3)} \int d^dx\sqrt{g}\mathcal{A}\varphi^2_{(3)}, \tag{57}$$

where

$$\begin{aligned} \mathcal{A} &= (d-5)[W^2 J - 4W^{2\mu\nu}K_{\mu\nu}] + 4(d-5)(d-4)W^{\mu\alpha\nu\beta}K_{\mu\nu}K_{\alpha\beta} \\ &\quad + \frac{4}{3}(d-5)(d-4)(d-3)[J^3 - 3K^2 J + 2K^3]. \end{aligned} \tag{58}$$

We can now insert (12) in (29) for $k = 3$ and get

$$E_{(3)} = \frac{16}{3}W^3 + \frac{32}{3}W^{\tilde{3}} + 8\mathcal{A}, \tag{59}$$

$$W^3 = W^{\alpha\beta}_{\mu\nu}W^{\mu\nu}_{\lambda\rho}W^{\lambda\rho}_{\alpha\beta} \quad , W^{\tilde{3}} = W_{\alpha\mu\nu\beta}W^{\lambda\mu\nu\rho}W^{\alpha\ \ \beta}_{\ \lambda\rho}. \tag{60}$$

In the same way as in the $k = 2$ case we can add these two additional invariant actions with the appropriate coefficients:

$$S_{W^3} = \frac{1}{2}\int d^dx\sqrt{g}\frac{4\Delta_3(W^3 + 2W^{\tilde{3}})}{(d-5)(d-4)(d-3)}\varphi^2_{(3)}, \tag{61}$$

and restore the Euler density containing Lagrangian

$$\begin{aligned} S^{E_{(3)}}_{(3)} &= S^{\mathcal{A}}_{(3)} + S_{W^3} \\ &= \frac{1}{2}\int d^dx\sqrt{g}\left\{-\varphi_{(2)}\Box^3\varphi_{(2)} + \ \dots\ + \frac{3\Delta_3}{4(d-5)(d-4)(d-3)}E_{(3)}\varphi^2_{(3)}\right\}, \end{aligned} \tag{62}$$

where we put $\dots$ instead of the other terms with derivatives, or terms proportional to $\Delta^2_{(3)}$ and $\Delta^3_{(3)}$. These terms can be readily read off (42), (44), (51) and (53).

We have proved our assertion concerning the connection between the hierarchy of conformally coupled scalars with the dimensions Δ_k and Euler densities $E_{(k)}$ for the $k = 1, 2, 3$, and have constructed the conformal coupling of the third scalar with gravity in dimensions $d \geq 6$. This action in spacetime dimension $d = 6$ or equivalently for $\Delta_{(3)} = 0$ degenerates to a conformal invariant operator for dimension 0 scalars obtained in [17, 18] from cohomological considerations of the effective action.

Conclusion

In conclusion, we formulate our assertion for general k case. Comparing (32), (41) and (62) we expect the following terms in the action of conformally coupled scalar with the scaling dimensions $\Delta_k = k - d/2$

$$S^{E_{(k)}}_{(k)} = \frac{(-1)^k}{2} \int d^d x \sqrt{g} \left\{ \varphi_{(k)} \Box^k \varphi_{(k)} + \ \dots \ - \frac{k!(d-2k)!\Delta_k}{2^k (d-k)!} E_{(k)} \varphi^2_{(k)} \right\} . \quad (63)$$

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